

Gauss sum factorization with cold atoms

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(Dated: Received: date / Revised version: date)

We report the first implementation of a Gauss sum factorization algorithm by an internal state Ramsey interferometer using cold atoms. A sequence of appropriately designed light pulses interacts with an ensemble of cold rubidium atoms. The final population in the involved atomic levels determines a Gauss sum. With this technique we factor the number $N=263193$.

The Shor algorithm [1] to factor numbers and its NMR-implementation [2] have propelled the field of quantum computation [3]. Recently a different factorization scheme [4] taking advantage of the periodicity properties of Gauss sums [5] has been proposed [6] and verified by two NMR-experiments [7, 8] and one experiment based on short laser pulses [9]. In the present paper we report the first implementation of Gauss sum factorization based on matter-wave interferometry [10] with cold rubidium atoms and use it to find the factors of $N = 263193$.

Our method rests on the observation that the truncated Gauss sum

$$C_N^{(M)}(l) = \frac{1}{M+1} \sum_{m=0}^M \cos\left(2\pi m^2 \frac{N}{l}\right) \quad (1)$$

consisting of $M+1$ terms yields unity if the integer l is a factor of N . In contrast, for integer non-factors l destructive interference leads to a small value of $C_N^{(M)}(l)$. Thus the algorithm checks if a given integer l is a factor of N or not. This testing is performed by a quantum system and its time evolution determines if a trial factor is a factor or not. Since in the worst case we have to test all prime numbers up to \sqrt{N} the prime number theorem [5] predicts the upper bound $\sqrt{N}/\log N$ for the number of trials. As a consequence our method in the present form scales exponentially which is not surprising since it does not involve entanglement yet and relies solely on interference. In this respect, our technique is very much in line with the recent critical discussion [11] of the connection between quantum mechanics and factorizing.

Our procedure for implementing Gauss sums is reminiscent of the method used in Ref. [7]. Common to both techniques is a sequence of pulses which imprints on a two-level quantum system a sequence of well-defined phases. For appropriately chosen pulses the excitation probability takes the form of a Gauss sum.

Despite these similarities our technique is different in many aspects: (i) In contrast to the NMR-approaches [7, 8] we do not use liquids but an ensemble of cold atoms. The spectacular control of internal as well as external atomic degrees of freedom with the help of lasers, opens up a new avenue towards factorization. Indeed, rubidium

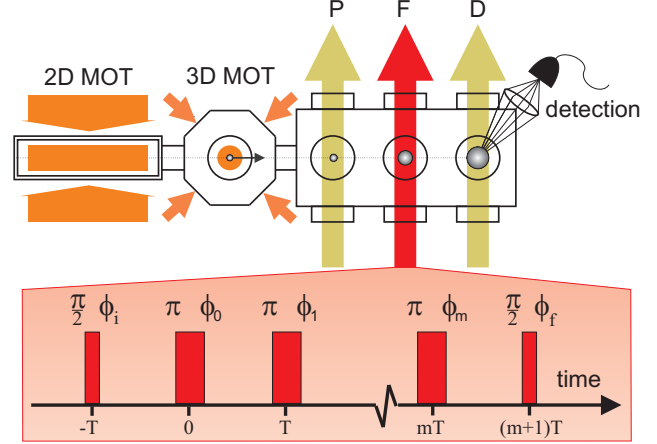


FIG. 1: Implementation of the Gauss sum factorization algorithm using cold rubidium atoms. They are launched by a double MOT system and prepared in the region P by an appropriate pulse sequence in the atomic ground state. In the factorization zone F the atoms interact with a sequence of pulses driving a hyperfine transition. We start with a $\pi/2$ -pulse followed by a sequence of $m+1$ π -pulses with the phase $\phi_k(l)$ and conclude by another $\pi/2$ -pulse. The phase during the $\pi/2$ -pulses is $\phi_i = \phi_f = -90^\circ$. A fluorescence detection in the region D measures the populations in both states which determines the interference signal $c_m(l)$. We record c_m for $0 \leq m \leq M$ and the sum over c_m yields the Gauss sum $C_N^{(M)}(l)$.

atoms feature two long-living hyperfine ground states which can be coherently manipulated by a two-photon Raman-transition [12]. Moreover, the use of cold atoms provides us with long interaction times and the possibility of a large number of pulses. (ii) In our approach there are no projection measurements between the pulses [7]. The measurement takes place after completing the sequence of pulses. (iii) The Gauss sum results from the addition of the measured signals obtained from an increasing number of pulses. As a consequence our resources in pulses scale quadratically whereas in the NMR-approach they only scale linearly. Despite this unfavorable scaling of the present technique we maintain that cold atoms offer several advantages. Most importantly, they have already proven to be ideal objects to be entangled. In particu-

lar, two-qubit quantum gates relying on cold atoms in standing light waves [13] have been realized. Therefore, we consider our experiment as a stepping stone towards more complex arrangements including entangled quantum systems.

Our experimental setup shown in Fig. 1 is part of an atom interferometer [14] designed to measure rotations with a high precision. The source of the cold Rb-atoms is a double stage MOT [15, 16]. A two-dimensional trap creates an atomic beam with a flux of 5×10^9 at/s for the efficient loading of a subsequent 3D-MOT. Approximately $N_{at} = 10^8$ trapped atoms are launched in a moving molasses, similar to atomic fountains [17] and are transferred into the atomic ground state $|5^2S_{1/2}, F = 1, m_F = 0\rangle$ using a multi-stage preparation.

In order to implement the Gauss sum, we use the pulse sequence illustrated in Fig. 1. An initial $\pi/2$ -pulse with phase $\phi_i = -90^\circ$ prepares a coherent superposition of ground and excited state. The latter corresponds to the hyperfine ground state $|5^2S_{1/2}, F = 2, m_F = 0\rangle$. This superposition is most sensitive to the phases of π -pulses. Indeed, after a time T we apply the factorization sequence consisting of $m + 1$ π -pulses each separated by the time T . The k -th pulse has the phase [7]

$$\phi_k(l) \equiv \begin{cases} 0 & \text{for } k = 0 \\ a_k(N)/l & \text{for } 1 \leq k \end{cases} \quad (2)$$

with $a_k(N) \equiv (-1)^k \pi N(2k - 1)$ which induces a coherent evolution of the atomic ensemble. We conclude at the time $(m + 1)T$ the factorization sequence by a $\pi/2$ -pulse with phase $\phi_f = -90^\circ$ to convert the phase evolution into a population difference between the two atomic states which is measured by a state-selective fluorescence detection [18].

The populations in the two states are governed [4, 7] by the interference signal

$$c_m(l) \equiv \cos\left(2\pi m^2 \frac{N}{l}\right) \quad (3)$$

which assumes values between -1 and +1 corresponding to all atoms in the ground or excited state, respectively. When we repeat the pulse sequence for increasing m with $0 \leq m \leq M$ and add the interference signals we arrive after normalization at the total signal $\mathcal{C}_N^{(M)}(l)$ determined by Eq. (1).

The multi-pulse excitation is executed by two digitally phase-locked Raman-lasers [19] which drive the hyperfine transition at approximately 6.834 GHz. The two beams are co-propagating through the factorization zone perpendicular to the trajectories of the atoms. In contrast to the velocity-sensitive excitation in inertial atomic sensors [20], in the present velocity-insensitive configuration we can neglect phase contributions from inertial forces.

A low phase noise oscillator serves as the reference for the phase locking setup. The phase $\phi_k(l)$ determined in

advance by a computer for each trial factor l is adjusted electronically by a synthesizer. The stringent requirements on the phase control in atom interferometry make the realization of the factorization experiment possible. The total phase error of the Raman-laser system is approximately 1 mrad. The length of a π -pulse is approximately $23 \mu\text{s}$ while the time T between two π -pulses is $100 \mu\text{s}$ which is sufficient for the electronic control loop to adjust the required phase in the laser system.

We now turn to the discussion of the results of our factorization experiment exemplified by the number $N = 263193 = 3 \times 7 \times 83 \times 151$. Figure 2 brings out most clearly that the dependence of the interference signal $c_m(l)$ on m is dramatically different for a factor of N such as $l = 151$ and a non-factor such as $l = 150$. Since for a factor $c_m(l)$ is approximately constant as a function of m the constructive addition of the terms in the definition Eq. (1) of the Gauss sum leads to a total signal $\mathcal{C}_N^{(M)}(l)$ close to unity. On the other hand a non-factor produces an oscillatory function $c_m(l)$ which takes on values between -1 and +1. Thus the total signal $\mathcal{C}_N^{(M)}(l)$ for a non-factor is rather small.

Figure 2 also shows a decay of the interference signal with increasing m . This behavior originates from the fact, that the atomic cloud on its trajectory sees a Gaussian intensity distribution of the Raman-laser beams which translates into a Gaussian distribution of the lengths of the $\pi/2$ - and π -pulses. This leads to a reduction of the transition probability and to a decrease of

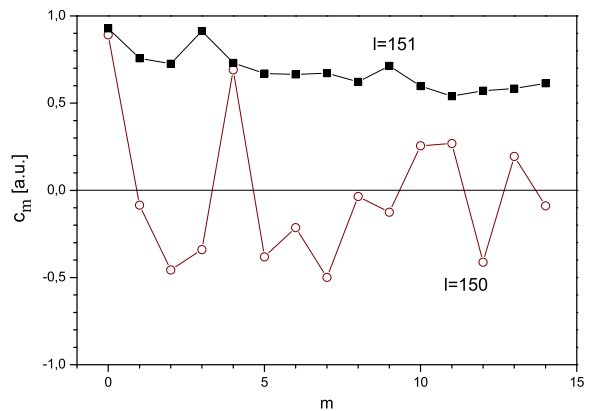


FIG. 2: Measured interference signals $c_m(l)$ as a function of the summation index m of the Gauss sum Eq. (1) determined by the number $m + 1$ of pulses in the factorization sequence for $N = 263193 = 3 \times 7 \times 83 \times 151$. Whereas $c_m(l)$ is approximately constant for the factor $l=151$ leading to the value $\mathcal{C}_N^{(14)}(151)=0.69$ the corresponding signal for the non-factor $l=150$ oscillates creating the small value $\mathcal{C}_N^{(14)}(150)=-0.02$. Here we have used the maximum number $M + 1 = 15$ of pulses.

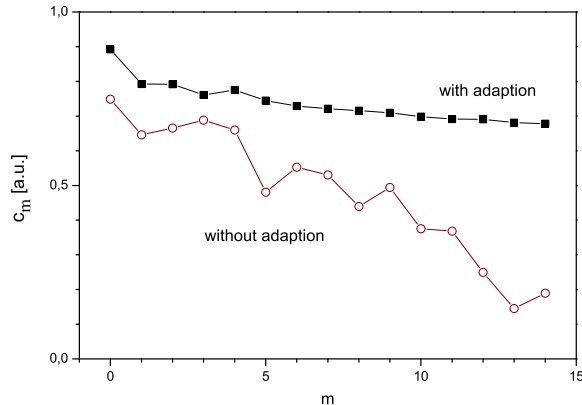


FIG. 3: Average of the interference signals $c_m(l)$ corresponding to three factors of $N = 263193$ as a function of m with and without a parabolic adaption of the pulse length.

the interference signal during the multi-pulse sequence.

We have compensated this disturbance by appropriately adjusting the pulse length over the total interaction region. For this purpose we have measured the lengths of the π -pulses for atoms in the center and in the edge of the interaction region which is $20 \mu s$ and $26 \mu s$, respectively. Since we can approximate a Gaussian in the neighborhood of its center by a polynomial of second degree we can connect the two measured pulse lengths by a parabola which provides us with an approximation of the pulse lengths over the whole interferometer region.

In Fig. 3 we compare the interference signals with and without such a pulse length adaption in their dependence on m . For this purpose we concentrate on factors only. In order to obtain a smooth curve we take the average of three such curves each corresponding to a different factor. This procedure clearly shows that with the adaption the decay is slowed down and consequently we can apply more pulses.

The remaining decay of $c_m(l)$ results from the interaction of the atomic cloud which has currently a diameter of about 5 mm in the interaction region with the Gaussian intensity distributed Raman-lasers leading to a reduction of the transition probability during each single pulse. Thus lower temperatures of the atoms would slow down the decay of $c_m(l)$.

An upper limit for the number $M + 1$ of possible pulses in the factorization sequence is experimentally set by the total interaction time $(M + 2)T$ which results from the non-vanishing velocity $v_{at} \sim 4.4$ m/s of the atoms propagating through the interaction region given by the diameter $d \sim 30$ mm of the Raman-laser beams. Thus we can apply up to 20 pulses without any excessively disturbing effects.

Figure 4 demonstrates the successful implementation of a Gauss sum factorization algorithm by an internal

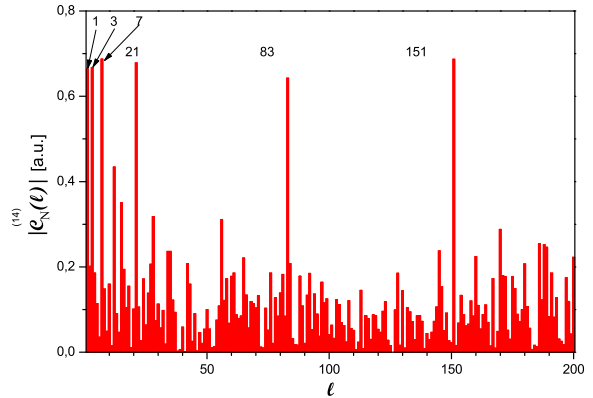


FIG. 4: Experimental Gauss sum factorization based on cold atoms and exemplified by the number $N = 263193 = 3 \times 7 \times 83 \times 151$ using the maximum number $M + 1 = 15$ of pulses. Here we display the absolute value of the sum $C_N^{(14)}(l)$ over the interference signals $c_m(l)$, that is the measured Gauss sum as a function of the trial factors $1 \leq l \leq 200$. Dominant peaks correspond to factors of N . The background is produced by the non-factors.

state Ramsey interferometer using cold atoms. Here we display the absolute value of the sum $C_N^{(M)}(l)$ of the interference signals $c_m(l)$ for all analyzed integer trial factors l for the number $N = 263193 = 3 \times 7 \times 83 \times 151$ recorded with the maximum number $M + 1 = 15$ of pulses. The corresponding factors of N stand out clearly from the background created by the non-factors. We also obtain factor-like signals for products of factors as exemplified by $l=21$, in complete agreement with the Gauss sum, Eq. (1).

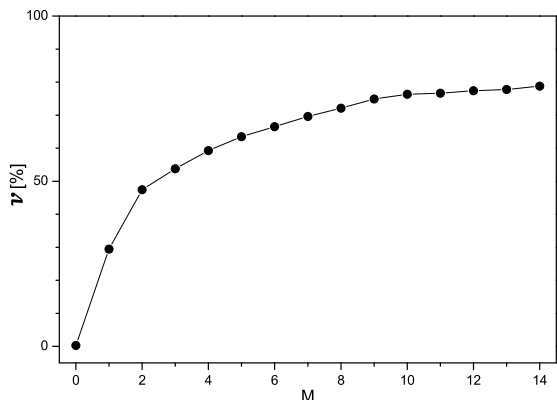


FIG. 5: Measured contrast \mathcal{V} of the factorization pattern for $N=263193$ as a function of M . Here $M + 1$ is the maximum number of factorization pulses. Already six pulses ($M = 5$) yield a contrast larger than 60%.

Figure 5 displays the experimentally obtained contrast \mathcal{V} of a factorization pattern such as the one in Fig. 4 as a function of the number M . Following Ref. [7] we have defined \mathcal{V} as the ratio between the difference and the sum of the measured averaged absolute values of the Gauss sum $\mathcal{C}_N^{(M)}(l)$ at factors and non-factors. We find that with just six pulses corresponding to $M = 5$ the number $N = 263193$ can be factored with a contrast larger than 60%, in agreement with the theoretical prediction of Ref. [7].

We have demonstrated the first implementation of a Gauss sum algorithm to factor numbers based on cold atoms and have successfully factorized the number $N=263193$. Our results are comparable to those of the NMR-experiments [7, 8]. However, the use of cold atoms not only represents an alternative approach but also allows us to envision several extensions: (i) Quadratic phases can be generated without prior calculation by a computer by a linear sweep of an external magnetic or electric field during the multi-pulse sequence [4]. (ii) The preparation of the atomic ensemble in an optical lattice opens up the possibility of applying an almost arbitrarily large number of pulses. (iii) A Gauss sum factorization scheme involving entanglement could rely on a large number of entangled atoms each one located in the minima of an optical lattice [21] providing us with a massive parallelism. The experiment reported in the present paper is the first step in these directions.

We appreciate stimulating discussions with B. Girard, D. Haase and M. Štefáňák. The work is supported by the SFB 407 of the Deutsche Forschungsgemeinschaft and the FINAQS cooperation of the European Union. One of us (WPS) is grateful to the Max-Planck Society and the Alexander von Humboldt Stiftung for their support. Moreover, he acknowledges the support by the Ministerium für Wissenschaft und Kunst, Baden-Württemberg and the Landesstiftung Baden-Württemberg in the framework of the Quantum Information Highway A8 and the Center for Quantum Engineering.

[1] P. Shor, in: *Proceedings of the 35th Annual Symposium on Foundations of Computer Science* (IEEE Computer Society, New York, 1994).
[2] For the implementation of the Shor algorithm with seven qubits factoring the number $15=3 \times 5$, see L.M.K. Vandersypen, M. Steffen, G. Breyta, C.S. Yannoni, M.H. Sherwood, and I.L. Chuang, *Nature* (London) **414**, 883 (2001).
[3] For the original papers suggesting quantum computation we refer to R.P. Feynman, *Int. J. theor. Phys.* **21**, 467 (1982); D. Deutsch, *Proc. R. Soc. Lond. A* **400**, 97 (1985); P.A. Benioff, *Int. J. theor. Phys.* **21**, 177 (1982); a particularly clear exposition of this field is contained in S. Stenholm and K.-A. Suominen, *Quantum Approach to Informatics* (John Wiley, New York, 2005); M.A. Nielsen

and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[4] W. Merkel, I.Sh. Averbukh, B. Girard, G.G. Paulus, and W.P. Schleich, *Fortschr. Phys.* **54**, 856 (2006); for a slightly augmented version see W. Merkel, I.Sh. Averbukh, B. Girard, M. Mehring, G.G. Paulus, and W.P. Schleich, in: *Elements of Quantum Information* edited by W.P. Schleich and H. Walther (Wiley-VCH, Weinheim, 2007).
[5] See, for example, H. Maier, and W.P. Schleich, *Prime Numbers 101: A Primer on Number Theory* (Wiley-VCH, New York, 2007).
[6] The factorization schemes proposed by J.F. Clauser, and J.P. Dowling, *Phys. Rev. A* **53**, 4587 (1996) and W.G. Harter, *Phys. Rev. A* **64**, 012312 (2001) also rely on the periodicity properties of Gauss sums. However, their Gauss sums are different from but are related to the present ones by the Gauss reciprocity law.
[7] M. Mehring, K. Müller, I.Sh. Averbukh, W. Merkel, and W.P. Schleich, *Phys. Rev. Lett.* **98**, 120502 (2007).
[8] T.S. Mahesh, N. Rajendran, X. Peng, and D. Suter, *Phys. Rev. A* **75**, 062303 (2007).
[9] B. Girard has kindly informed us about his recent experiment with B. Chatel and D. Bigourd factorizing numbers with Gauss sums based on a sequence of femtosecond laser pulses generated by a pulse shaper. The two-level atom is replaced by a spectrometer tuned to a single wavelength. The pulses are delayed such that the relative phase of a given wavelength is the required phase for the individual terms of the Gauss sum.
[10] M. Kasevich and S. Chu, *Phys. Rev. Lett.* **67**, 181 (1991).
[11] N.D. Mermin, *Physics Today*, p.8 (April 2007); see also M.S. Zubairy, *Science* **316**, 554 (2007).
[12] M. Kasevich, D.S. Weiss, E. Riis, K. Moler, S. Kasapi, and S. Chu, *Phys. Rev. Lett.* **66**, 2297 (1991).
[13] O. Mandel, M. Greiner, A. Widera, T. Rom, T.W. Hänsch, and I. Bloch, *Nature* **425**, 937 (2003).
[14] C. Jentsch, T. Müller, E.M. Rasel, and W. Ertmer, *Gen. Rel. Grav.* **36**, 2193 (2004).
[15] E.L. Raab, M. Prentiss, A. Cable, S. Chu, and D.E. Pritchard, *Phys. Rev. Lett.* **59**, 2631 (1987).
[16] T. Müller, T. Wendrich, M. Gilowski, C. Jentsch, E.M. Rasel, and W. Ertmer, *physics.atom-ph*, arXiv:0705.4544v1.
[17] S. Bize, Y. Sortais, M.S. Santos, C. Mandache, A. Clairon, and C. Salomon, *Europhys. Lett.* **45**, 558 (1999).
[18] Y. Sortais, S. Bize, C. Nicolas, G. Santarelli, C. Salomon, and A. Clairon, *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control* **47**, 1093 (2000).
[19] For a detailed description of an optical phase-lock loop similar to the one implemented in our setup see G. Santarelli, A. Clairon, S.N. Lea, and G.M. Tino, *Opt. Comm.* **104**, 339 (1994) or L. Cacciapiuoti, M. de Angelis, M. Fattori, G. Lamporesi, T. Petelski, M. Prevedelli, J. Stuhler, and G.M. Tino, *Rev. Sci. Instrum.* **76**, 053111 (2005).
[20] B. Canuel, F. Leduc, D. Holleville, A. Gauguier, J. Fils, A. Virdis, A. Clairon, N. Dimarcq, Ch.J. Bordé, P. Bouyer and A. Landragin, *Phys. Rev. Lett.* **97**, 010402 (2006).
[21] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).